

GENERALIZED PUISIEUX SERIES EXPANSION FOR COSMOLOGICAL MILESTONES

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We use generalized Puiseux series expansions to determine the behaviour of the scale factor in the vicinity of typical cosmological milestones occurring in a FRW universe. We describe some of the consequences of this generalized Puiseux series expansion on other physical observables.

1. Introduction

Over the last few years, the zoo of cosmological singularities considered in the literature has been considerably expanded, with “big rips” and “sudden singularities” added to the “big bang” and “big crunch”, as well as renewed interest in non-singular cosmological events such as “bounces” and “turnarounds”.^{1–5}

We consider a cosmological spacetime of the FRW form and assume applicability of the Einstein equations of general relativity. We will provide a generic definition of all the physically relevant singularities considered above (which we shall refer to as cosmological milestones), using generalized Puiseux series for the scale factor of the universe $a(t)$. We will show that, most importantly, *all* physical observables (H , q , the Riemann tensor, *etc.*) will likewise be described by a generalized Puiseux series.

2. Generalized Puiseux series expansion of the scale factor $a(t)$

Solutions of differential equations can often be expanded in Taylor series or Laurent series around their singular points. We shall extend this idea by expanding the scale factor $a(t)$ in generalized power series, similar to a Puiseux series, in the vicinity of the cosmological milestones.

Generic cosmological milestone: Suppose we have some unspecified generic cosmological milestone, that is defined in terms of the behaviour of the scale factor $a(t)$, and which occurs at some finite time t_{\odot} . We will assume that in the vicinity of the milestone the scale factor has a (possibly one-sided) generalized power series

expansion of the form

$$a(t) = c_0|t - t_\odot|^{\eta_0} + c_1|t - t_\odot|^{\eta_1} + c_2|t - t_\odot|^{\eta_2} + c_3|t - t_\odot|^{\eta_3} + \dots \quad (1)$$

where the indicial exponents η_i are generically real (and are often non-integer) and without loss of generality are ordered in such a way that they satisfy

$$\eta_0 < \eta_1 < \eta_2 < \eta_3 \dots \quad (2)$$

Finally we can also without loss of generality set $c_0 > 0$. There are no *a priori* constraints on the signs of the other c_i , though by definition $c_i \neq 0$.

The first term of the right hand side of equation (1) is the dominant term, and is therefore responsible for the convergence or divergence of the scale factor at the time t_\odot . The indices η_i are used to classify the cosmological milestones and the absolute value symbols are used to distinguish a past event from a future event. This generalized power series expansion of the scale factor is sufficient to represent almost all the physical models that we are aware of in the literature. Table 1 represents this cosmological milestone classification depending on the value of the scale factor. Note that sudden singularities are of order n where the n^{th} derivative of the scale

Cosmological milestones	Scale factor value	Indices η_i
Big Bang/ Big Crunch	$a(t_\odot) = 0$	$\eta_0 > 0$
Sudden Singularity	$a(t_\odot) = c_0$ $a^{(n)}(t_\odot) = \infty$	$\eta_0 = 0$ η_1 non-integer
Extremality events	$a(t_\odot) = c_0$	$\eta_0 = 0$ $\eta_i \in \mathbb{Z}^+$
Big rip	$a(t_\odot) = \infty$	$\eta_0 < 0$

factor is the first one that is infinite:

$$a^{(n)}(t \rightarrow t_\odot) \sim c_0 \eta_1(\eta_1 - 1)(\eta_1 - 2) \dots (\eta_1 - n + 1) |t - t_\odot|^{\eta_1 - n} \rightarrow \infty, \quad (3)$$

and therefore η_1 has to be a non-integer.^{3,4} Note that for most calculations it is sufficient to use the first three (or fewer) terms of the power series expansion.

3. Power series expansion of all physical observables

We have exhibited a generic expansion of the scale factor $a(t)$ based on generalized power series for all the physically relevant cosmological milestones found in the literature to date (big bang, big crunch, sudden singularity, extremality events and big rip). We can now use the parameters of this series to explore the kinematical and dynamical properties of the cosmological milestones, for example, to see whether

they are true curvature singularities or whether the energy conditions hold in the vicinity of the time of the event t_\odot .

For instance, on a kinematical level, we can analyze the Hubble parameter for finiteness in the vicinity of the cosmological milestones. Keeping the most dominant terms, we have for $\eta_0 \neq 0$:

$$H = \frac{\dot{a}}{a} \sim \frac{c_0 \eta_0 (t - t_\odot)^{\eta_0 - 1}}{c_0 (t - t_\odot)^{\eta_0}} = \frac{\eta_0}{t - t_\odot}; \quad (\eta_0 \neq 0). \quad (4)$$

That is, for bangs, crunches, and rips the Hubble parameter exhibits a generic $1/(t - t_\odot)$ blow up.

In a similar fashion, we can also determine whether a cosmological milestone is a *true curvature singularity* by testing $R_{\hat{t}\hat{t}}$ and $G_{\hat{t}\hat{t}}$ in orthonormal components for finiteness:

$$R_{\hat{t}\hat{t}} = -3 \frac{\ddot{a}}{a}; \quad G_{\hat{t}\hat{t}} = 3 \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right). \quad (5)$$

On a dynamical level, we can quantify how “strange” physics gets in the vicinity of a cosmological milestone by introducing the Friedmann equations and the standard energy conditions in general relativity — which are the *null*, *weak*, *strong*, and *dominant* energy conditions.^{5–7} The density and pressure are given as a function of the scale factor $a(t)$ and can therefore likewise be power series. Whether or not a specific energy condition is satisfied is simply a matter of calculating the dominant indicial exponents of the series expansion (full details provided in⁸).

To conclude, if in the vicinity of any cosmological milestone, the input scale factor $a(t)$ is a generalized power series, then all physical observables (*e.g.* H , q , the Riemann tensor, *etc.*) will likewise be a generalized Puiseux series. By checking the related indicial exponents, which can be calculated from the indicial exponents of the scale factor, one can determine whether or not the particular physical observable then diverges at the cosmological milestone.

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